THE CACCETTA-HÄGGKVIST CONJECTURE

Adrian Bondy

What is a *beautiful* conjecture?

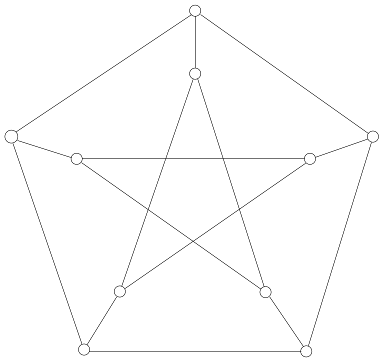
The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics.

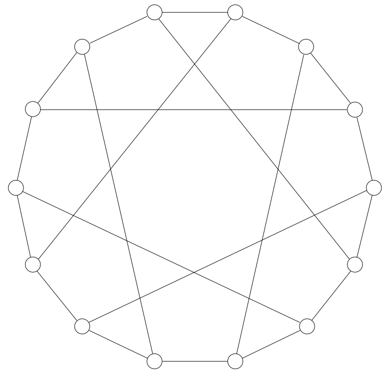
G.H. Hardy

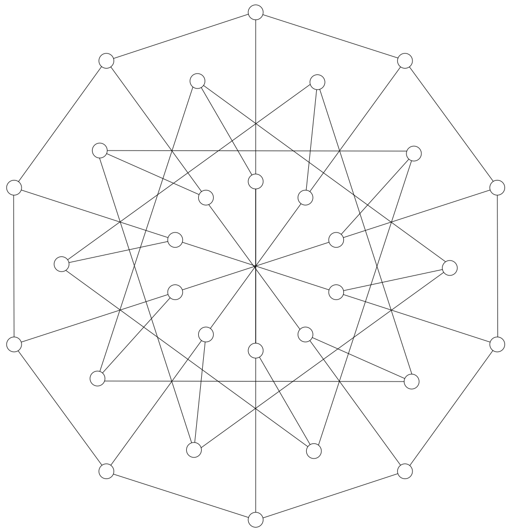
Some criteria:

- ▷ *Simplicity*: short, easily understandable statement relating basic concepts.
- ▷ *Element of Surprise*: links together seemingly disparate concepts.
- ▷ *Generality*: valid for a wide variety of objects.
- ▷ *Centrality*: close ties with a number of existing theorems and/or conjectures.
- ▷ *Longevity*: at least twenty years old.
- ▷ *Fecundity*: attempts to prove the conjecture have led to new concepts or new proof techniques.

(d, g)-*cage*: smallest d-regular graph of girth g







Lower bound on order of a (d, g)-cage:

girth $g = 2r$ order $\frac{2(d-1)^r - 2}{d-2}$

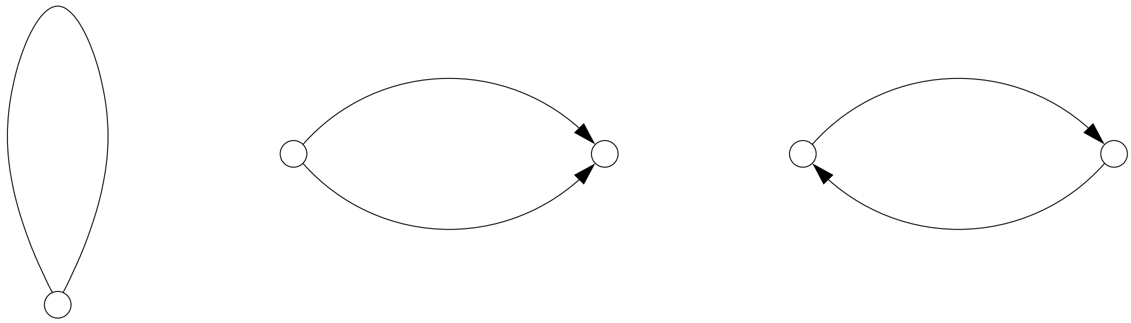
girth $g = 2r + 1$ order $\frac{d(d-1)^r - 2}{d-2}$

Examples with equality:

- $\triangleright$ Petersen
- $\triangleright$ Heawood
- $\triangleright$ Coxeter-Tutte
- $\triangleright$ Hoffman-Singleton ...

We shall consider only oriented graphs:

no loops, parallel arcs or directed 2-cycles

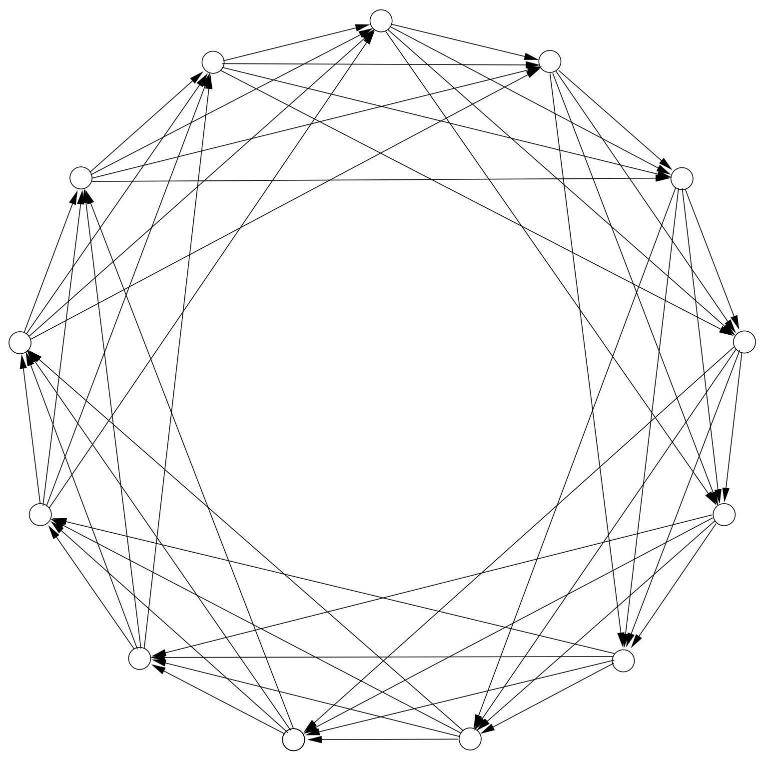


Directed (d, g)-*cage*:

smallest d-diregular digraph of directed girth g

Behzad-Chartrand-Wall Conjecture 1970

The digraph $\overrightarrow{C}^{d}_{d(g-1)+1}$ *is a directed* (d, g)-*cage*



DIRECTED $(4, 4)$-CAGE?

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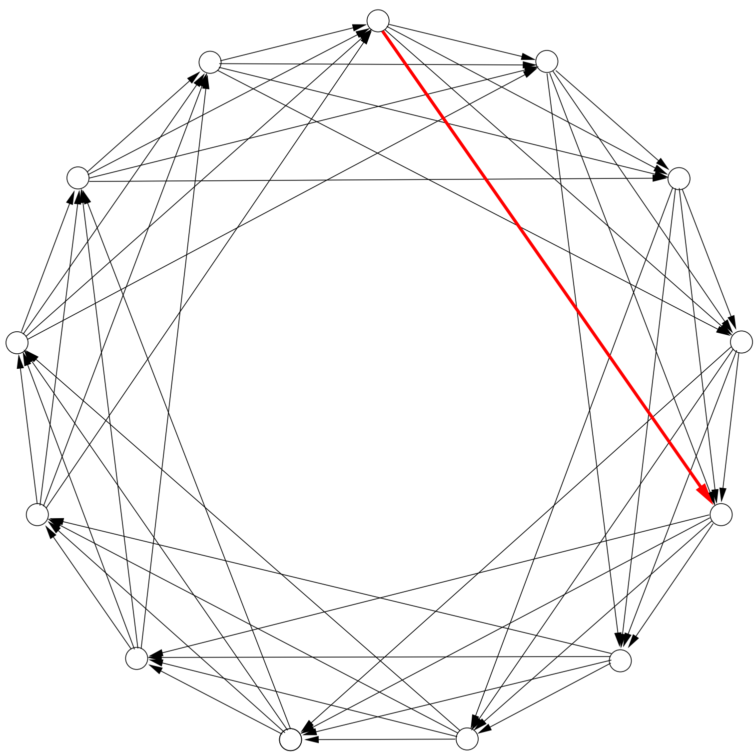

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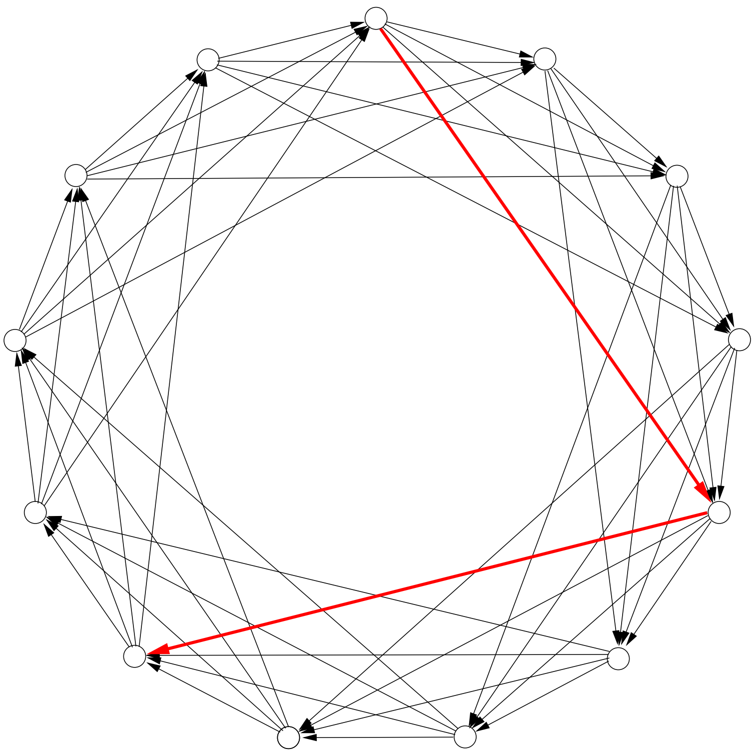

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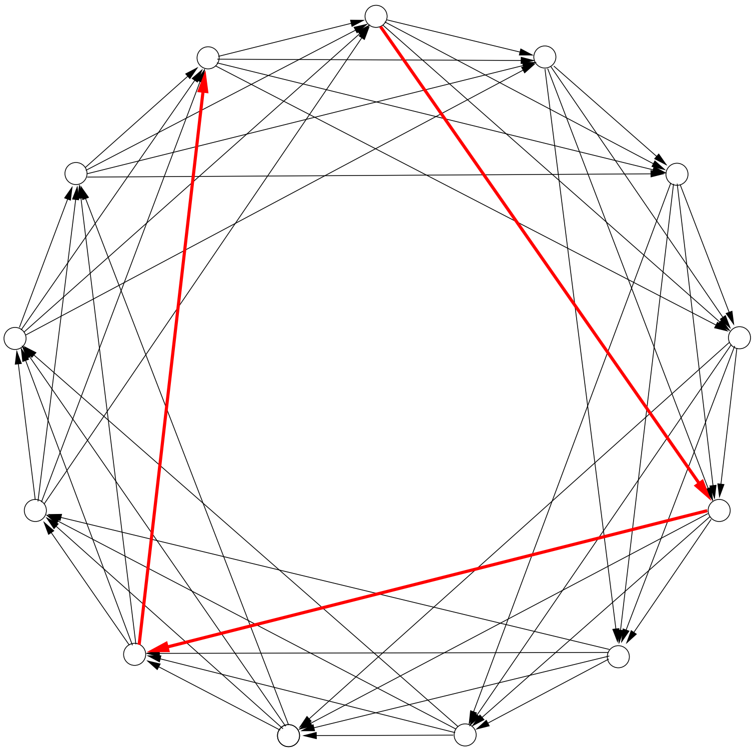

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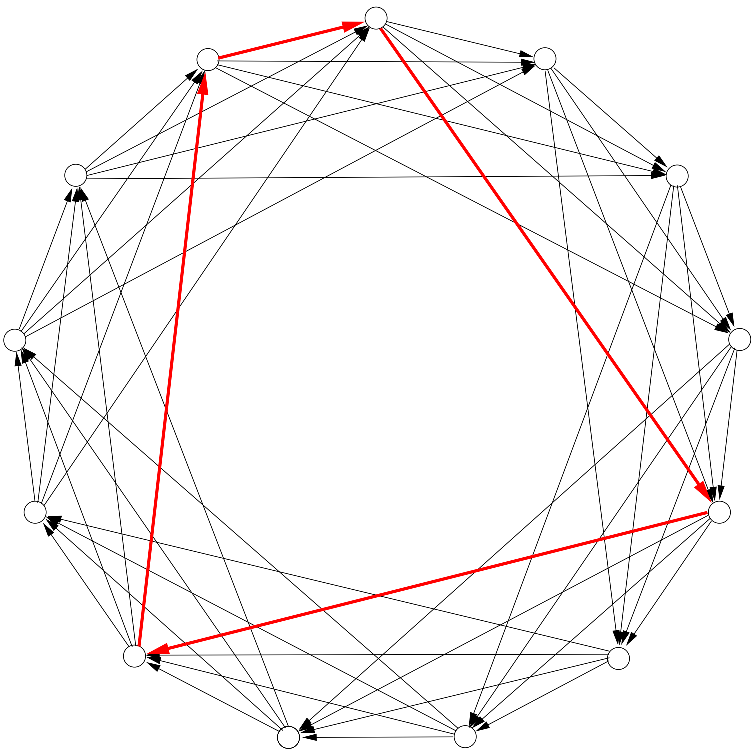
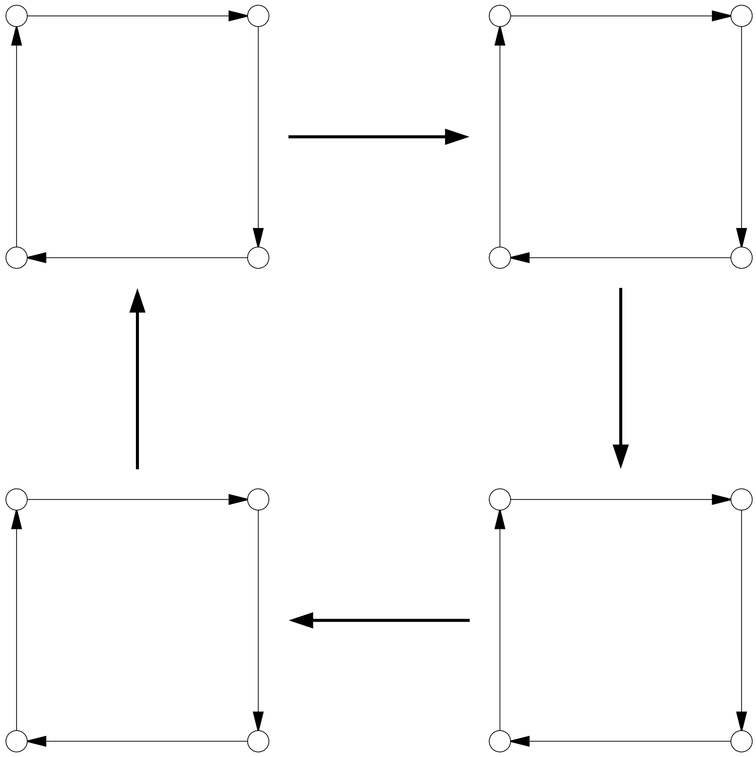


DIRECTED $(4, 4)$-CAGE?

COMPOSITIONS



DIRECTED $(5,4)$-CAGE?

More generally, if G and H are directed (d,g)-cages, then so is their composition $G[H]$

Reformulation:

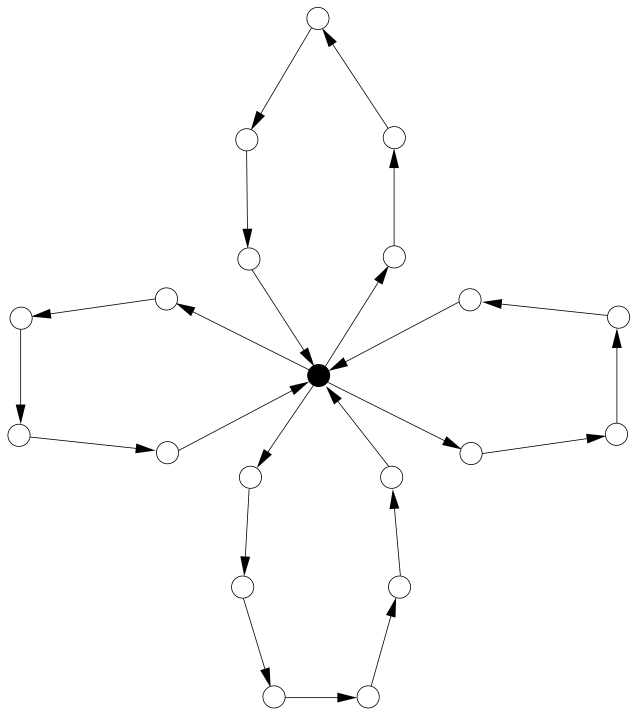
Behzad-Chartrand-Wall Conjecture 1970

Every d-diregular digraph on n vertices has a directed cycle of length at most $\lceil n/d \rceil$

VERTEX-TRANSITIVE GRAPHS

HAMIDOUNE:

In a d-diregular vertex-transitive digraph, there are d directed cycles $C_1, \ldots, C_d$ passing through a common vertex, any two meeting only in that vertex:



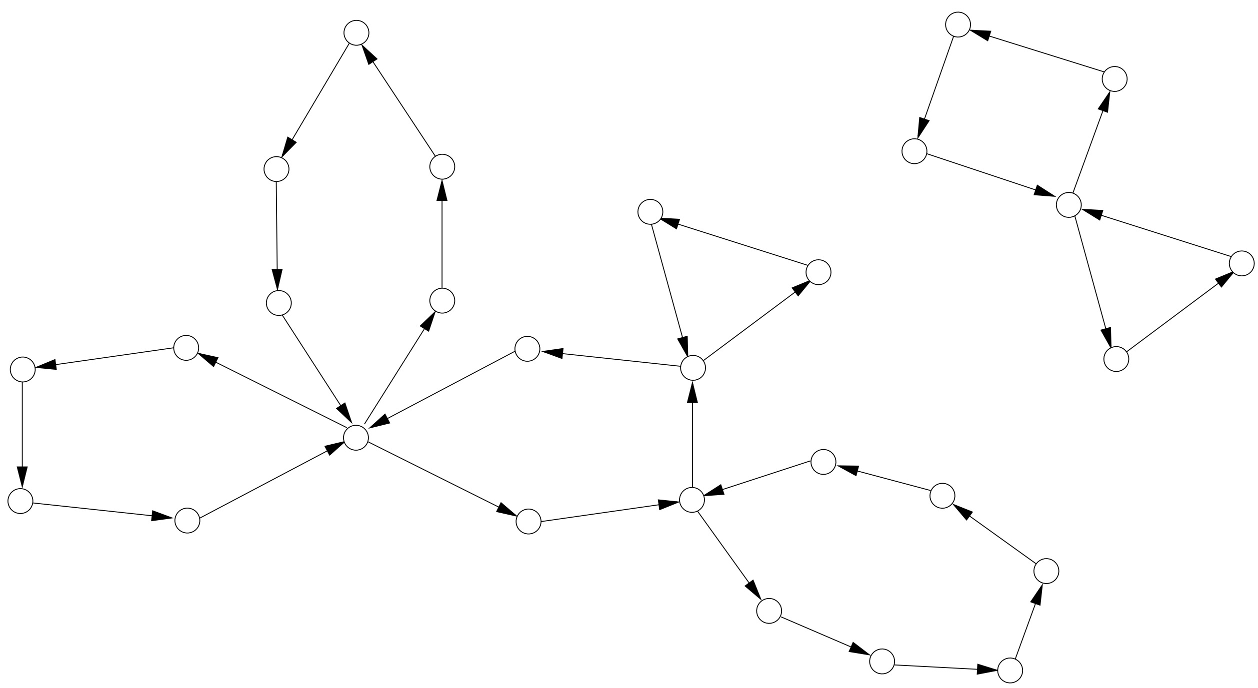
$$\sum_{i=1}^{d} |V(C_i)| \leq n + d - 1$$

So one of these cycles is of length at most $\left\lceil \frac{n}{d} \right\rceil$

DISJOINT DIRECTED CYCLES

Hoáng-Reed Conjecture 1987

In a d-diregular digraph, there are d directed cycles $C_1, \ldots, C_d$ such that C_j meets $\cup_{i=1}^{j-1} C_i$ in at most one vertex, $1 < j \leq d$.

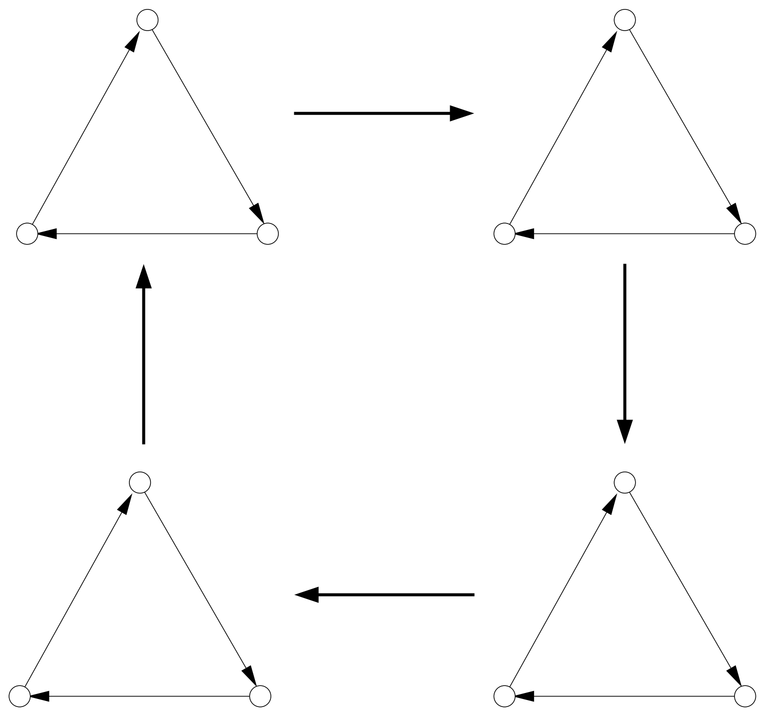


FOREST OF d DIRECTED CYCLES

Mader:

Forest of directed cycles not necessarily linear:

$$C_d[C_{d-1}]$$



No linear forest of four directed cycles

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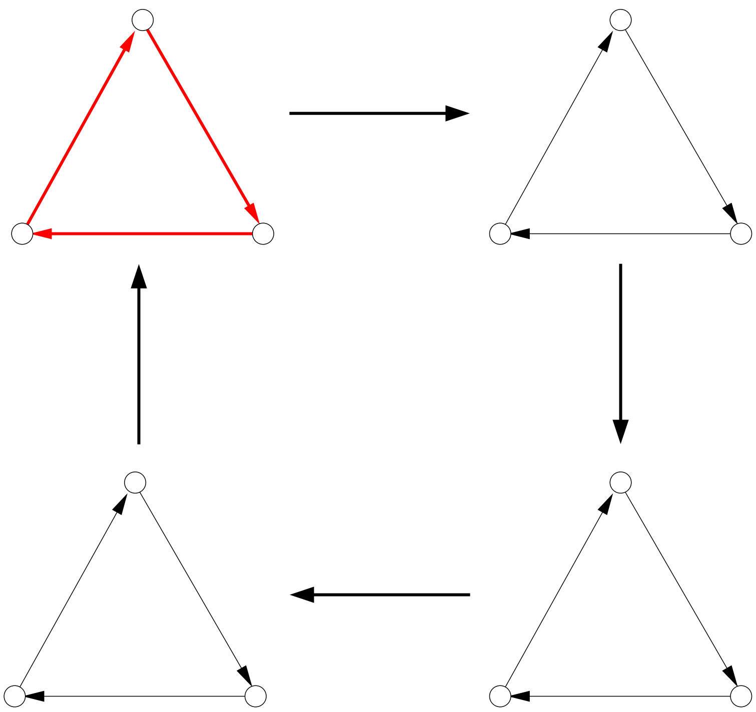

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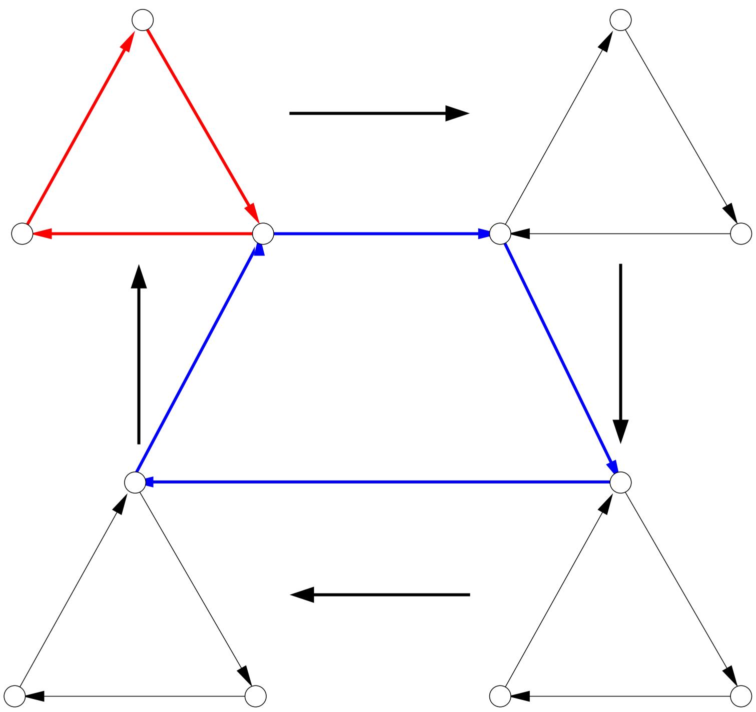


No linear forest of four directed cycles

PRESCRIBED MINIMUM OUTDEGREE

Caccetta-Häggkvist Conjecture 1978

Every digraph on n vertices with minimum outdegree d has a directed cycle of length at most $\lceil n/d \rceil$

WHAT IS KNOWN?

CACCETTA AND HÄGGKVIST: $d = 2$

HAMIDOUNE: $d = 3$

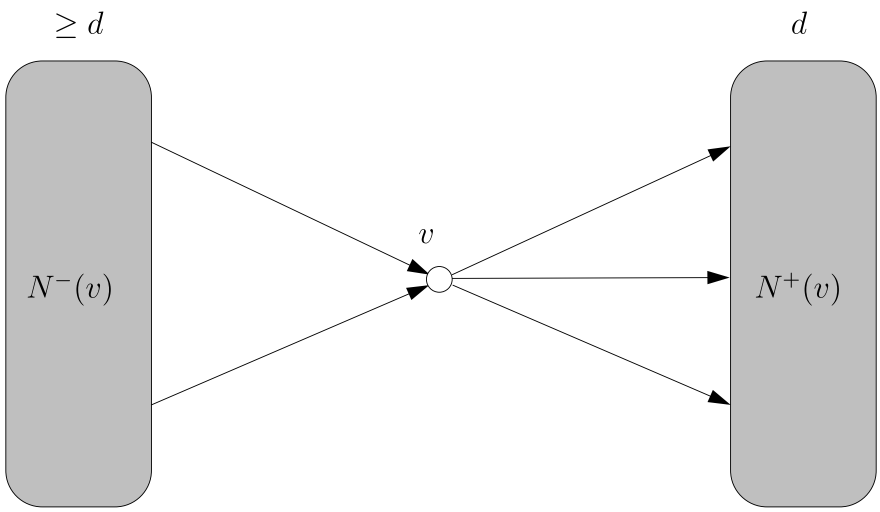
HOÁNG AND REED: $d = 4, 5$

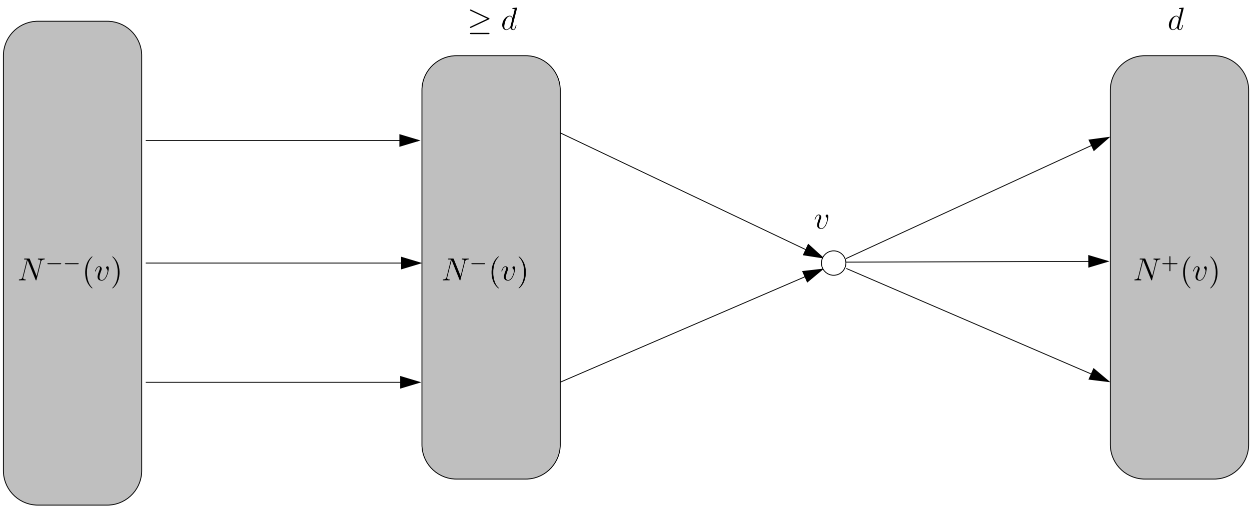
SHEN: $d \leq \sqrt{n/2}$

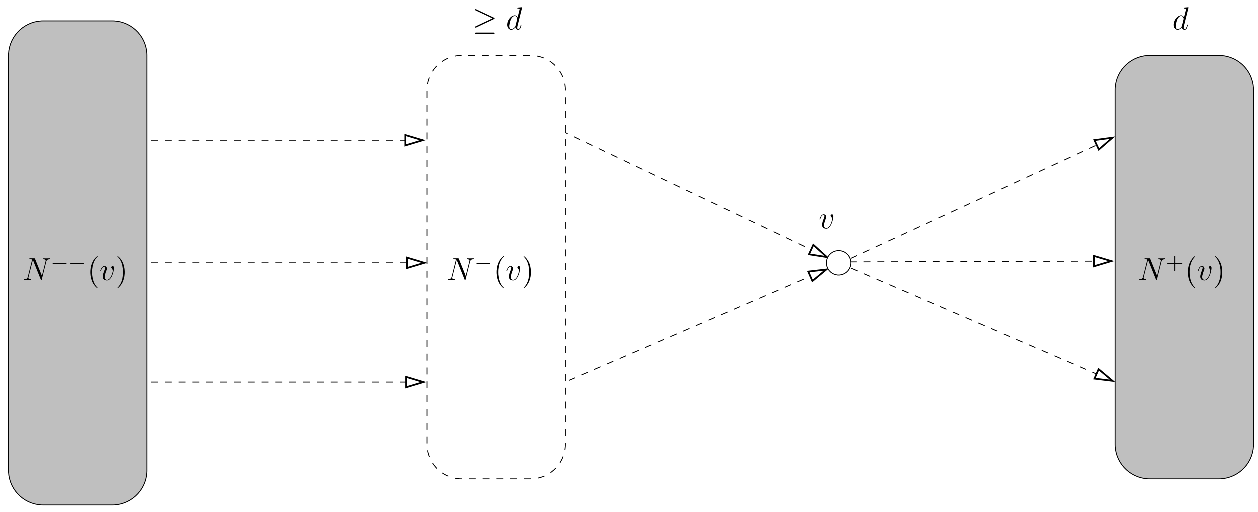
Chvátal and Szemerédi:

Every digraph on n vertices with minimum outdegree d has a directed cycle of length at most $2n/d$

Proof by Induction:




$\geq d$
d
$N^{--}(v)$
$N^{-}(v)$
v
$N^{+}(v)$



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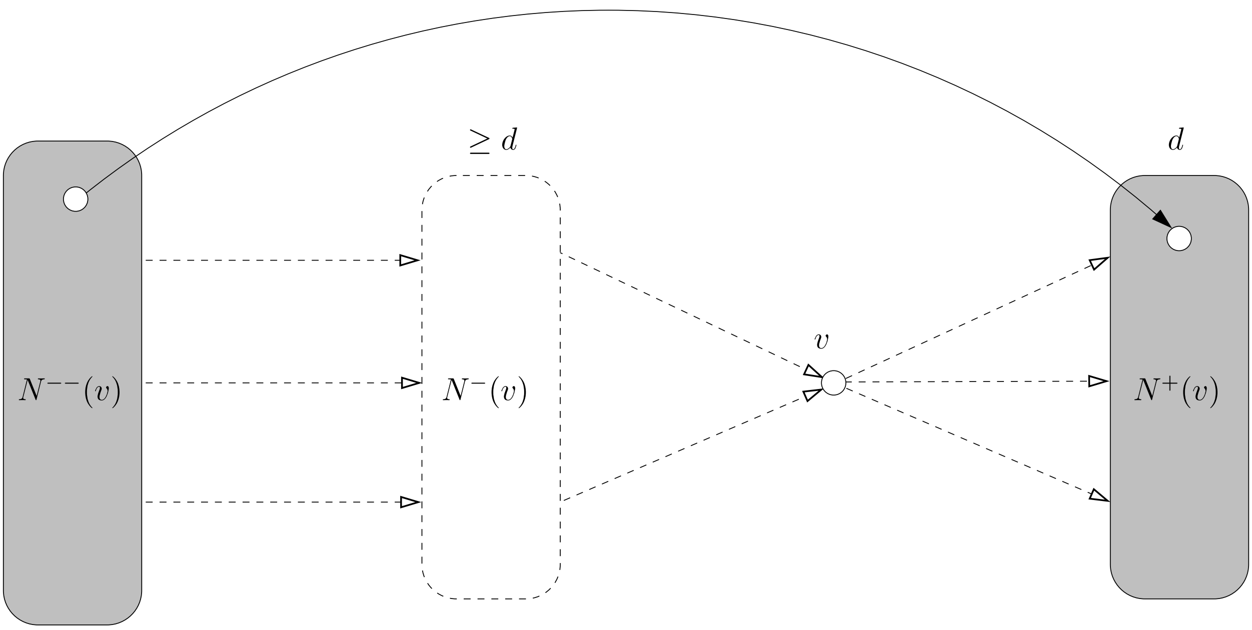

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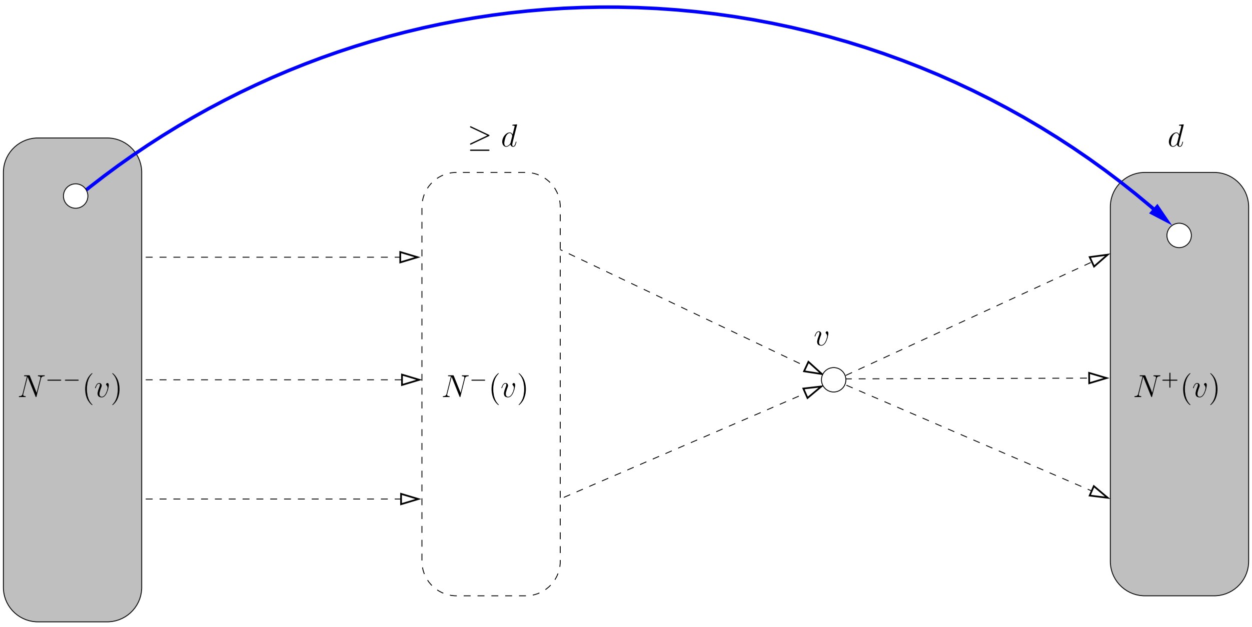
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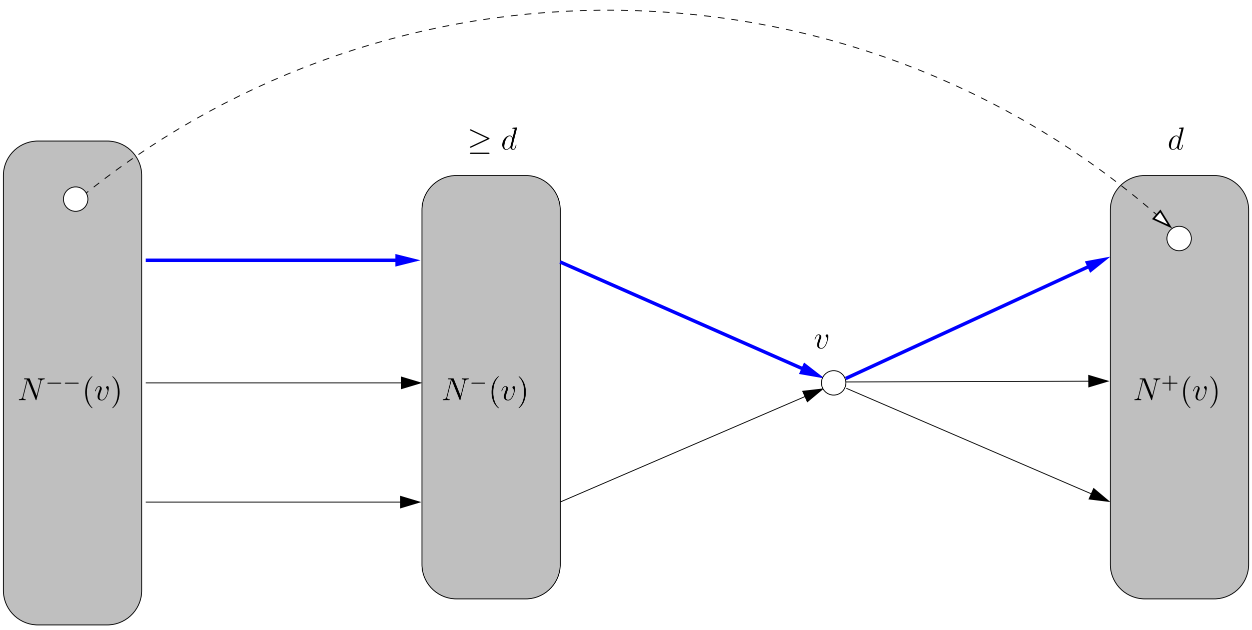
$\geq d$
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CHVÁTAL AND SZEMERÉDI:

Every digraph on n vertices with minimum outdegree d has a directed cycle of length at most $(n/d) + 2500$

SHEN:

Every digraph on n vertices with minimum outdegree d has a directed cycle of length at most $(n/d) + 73$

WHAT DOES THIS SAY WHEN $d = \lceil n/3 \rceil$?

Every digraph on n vertices with minimum outdegree $\lceil n/3 \rceil$ *has a directed cycle of length at most* 76

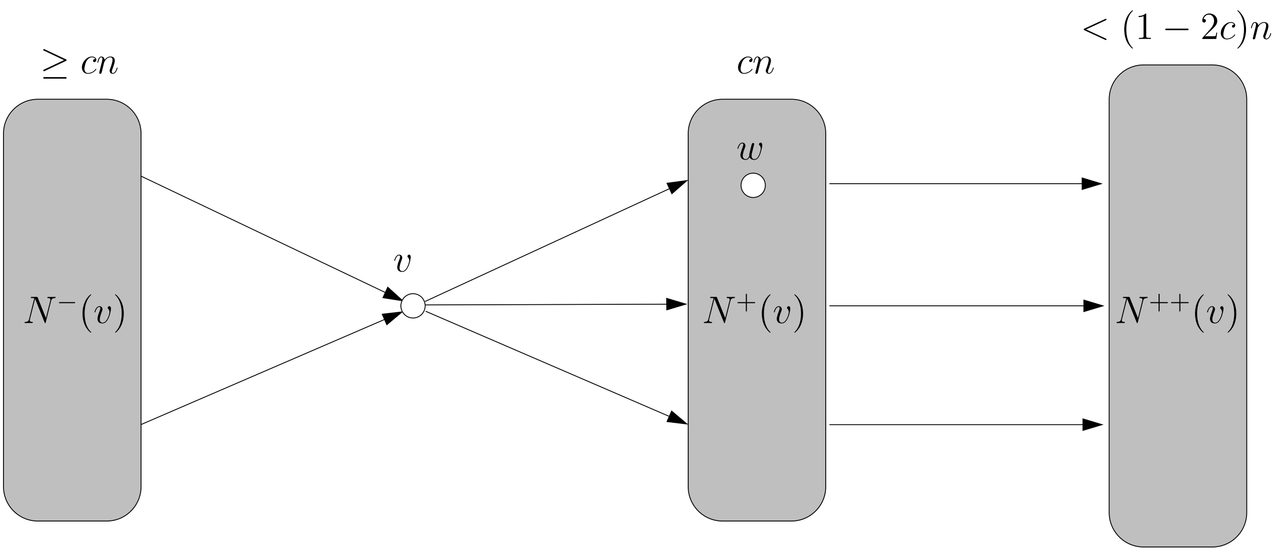
BUT THE BOUND IN THE CACCETTA-HÄGGKVIST CONJECTURE IS 3

Caccetta-Häggkvist Conjecture for triangles

Every digraph on n vertices with minimum outdegree $\lceil n/3 \rceil$ has a directed triangle

Caccetta and Häggkvist:

Every digraph on n vertices with minimum outdegree $\lceil cn \rceil$, where $c = \frac{1}{2}(3 - \sqrt{5})$, has a directed triangle



Assume no directed triangle.
Apply induction to subgraph induced by $N^+(v)$:

$$cn \leq d^+(w) < c^2n + (1-2c)n \quad \text{so} \quad c^2 - 3c + 1 > 0$$

DEGREE BOUNDS FOR A TRIANGLE

minimum outdegree $\lceil cn \rceil$:

Caccetta and Häggkvist: $c = \frac{1}{2}(3 - \sqrt{5}) \;\approx\; 0.382$

Shen: $c = 3 - \sqrt{7} \;\approx\; 0.3542$

minimum indegree and outdegree at least $\lceil cn \rceil$:

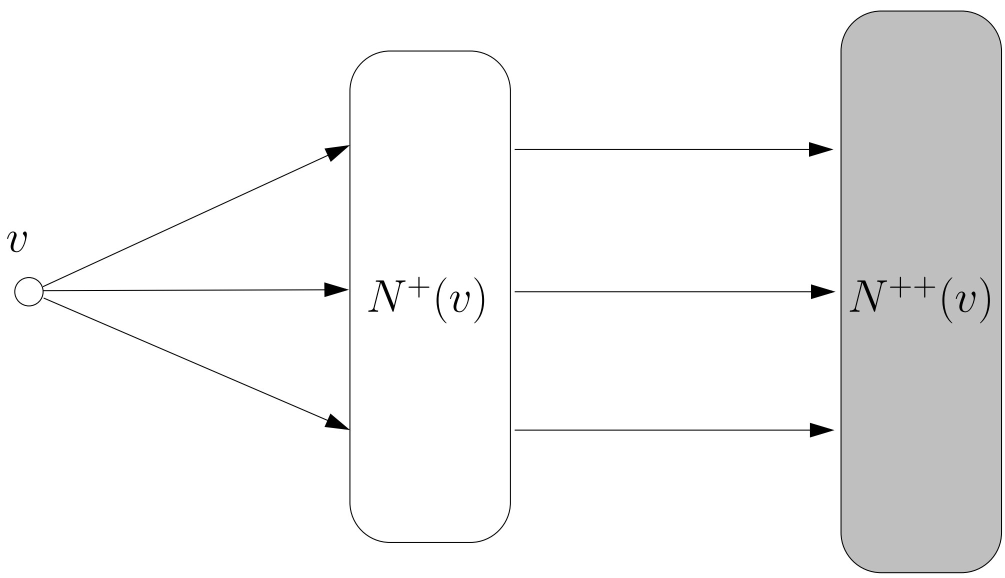
de Graaf, Seymour and Schrijver: $c \;\approx\; .3487$

Shen: $c \;\approx\; 0.3477$

SECOND NEIGHBOURHOODS

Seymour's Second Neighbourhood Conjecture 1990

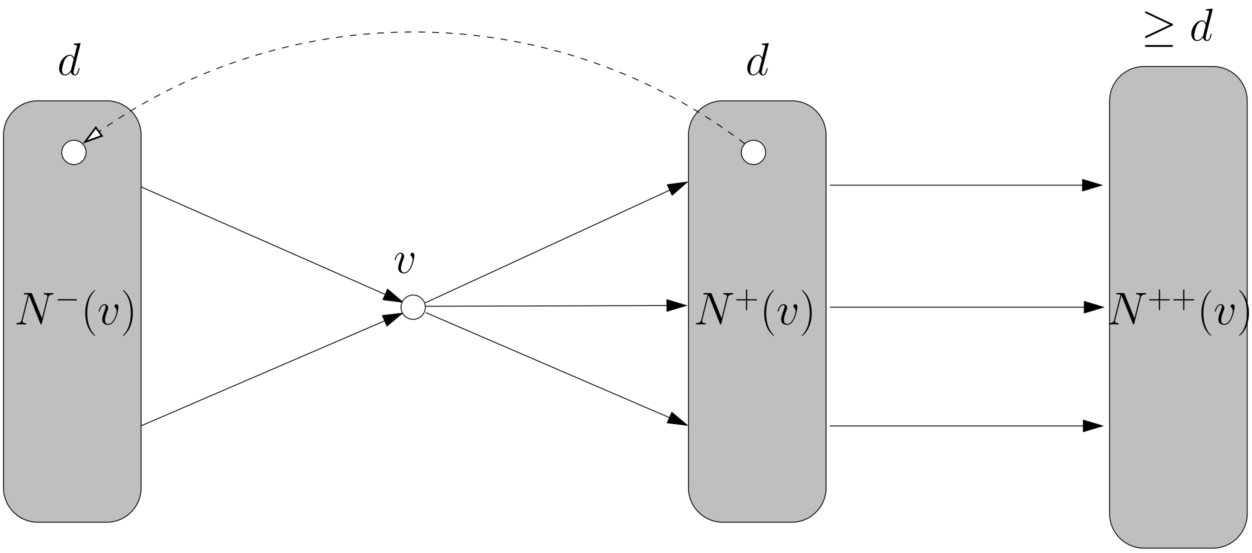
Every digraph (without directed 2-cycles) has a vertex with at least as many second neighbours as first neighbours



The *Second Neighbourhood Conjecture* implies the triangle case

$$d = \left\lceil \frac{n}{3} \right\rceil$$

of the *Behzad-Chartrand-Wall Conjecture*



If there is no directed triangle:

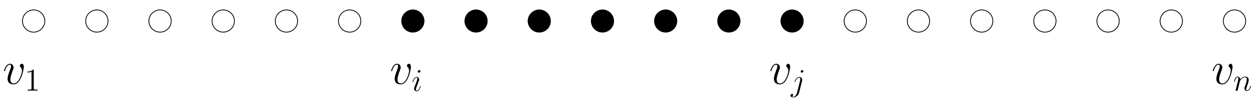
$$n \geq 3d + 1$$

Fisher: Second Neighbourhood Conjecture true for tournaments

Proof by Havet and Thomassé

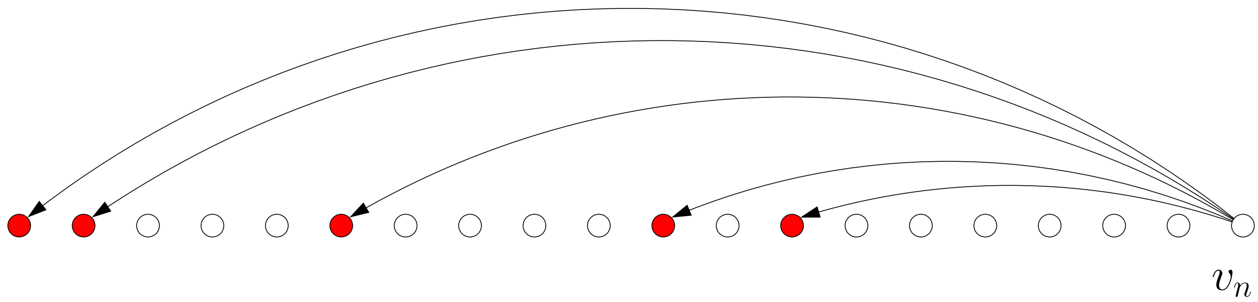
Median order: linear order $v_1, v_2, \ldots, v_n$ maximizing $|\{(v_i, v_j) : i < j\}|$ (number of arcs from left to right)

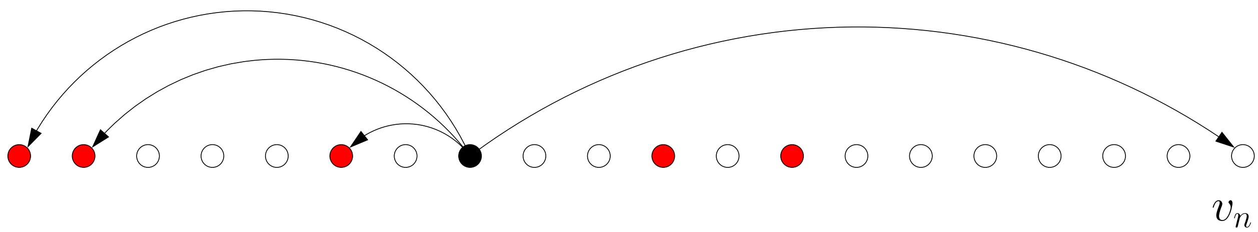
Property: for any $i \leq j$, vertex v_j is dominated by at least half of the vertices $v_i, v_{i+1}, \ldots, v_{j-1}$

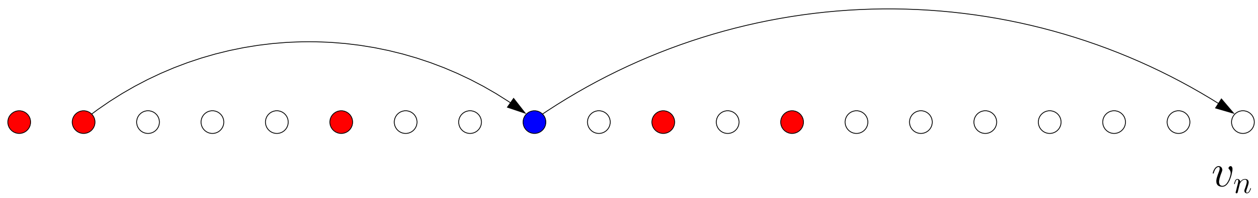


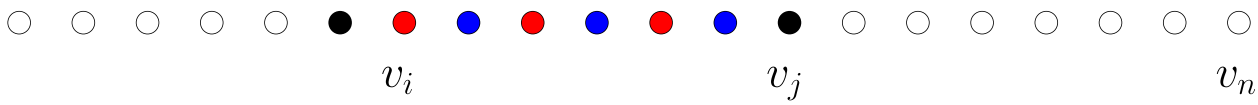
If not, move v_j before v_i

Claim: $|N^{++}(v_n)| \geq |N^+(v_n)|$


v_n


v_n


v_n


v_i
v_j
v_n

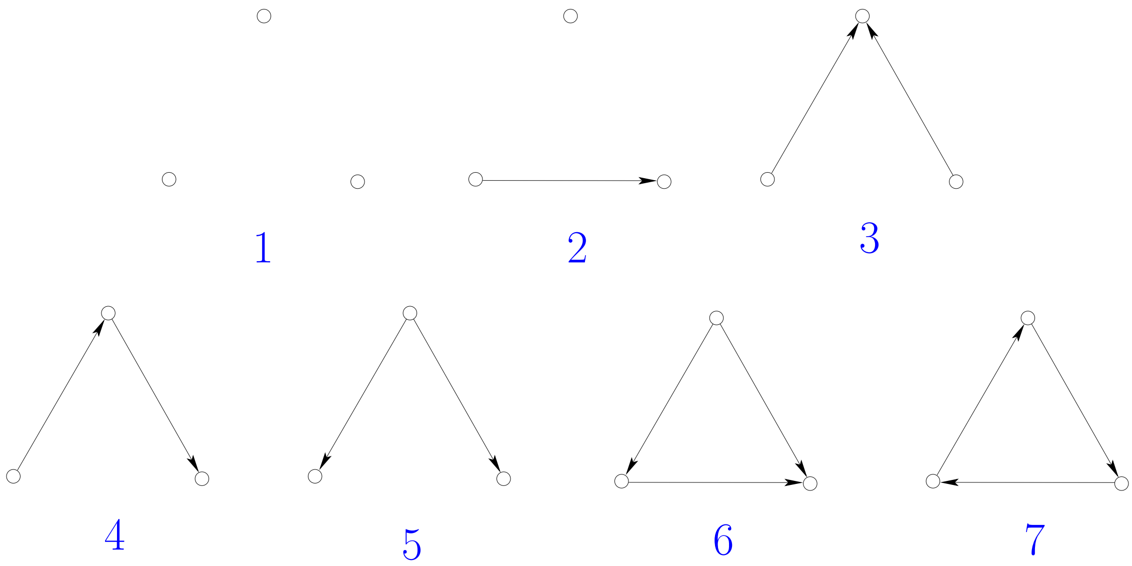
COUNTING SUBGRAPHS

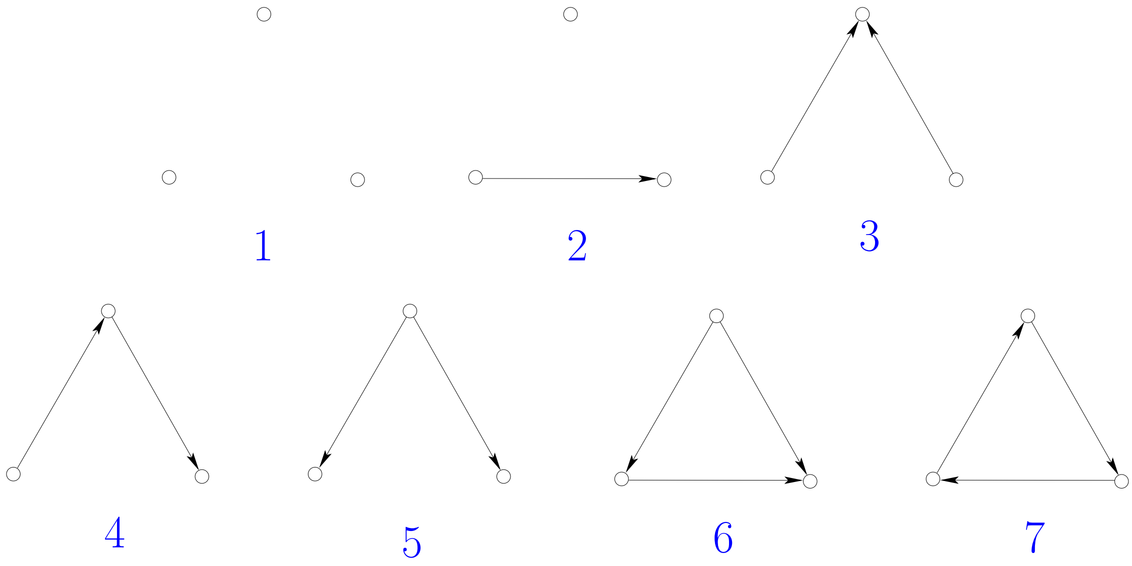
NOTATION

D digraph

$d^-(v)$ indegree of v, d outdegree of v, $v \in V$

Seven possible types of induced 3-vertex subgraphs:





x_i number of *induced* subgraphs of type i in D

$$
\begin{array}{ccccccccccccccc}
x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & + & x_7 & = & \binom{n}{3} \\
 & & x_2 & + & 2x_3 & + & 2x_4 & + & 2x_5 & + & 3x_6 & + & 3x_7 & = & n(n-2)d \\
 & & & & x_3 & & & & & + & x_6 & & & = & \sum_{v \in V} \binom{d^-(v)}{2} \\
 & & & & & & x_4 & & & + & x_6 & + & 3x_7 & = & nd^2 \\
 & & & & & & & & x_5 & + & x_6 & & & = & n\binom{d}{2}
\end{array}
$$

Assume no directed triangle: $x_7 = 0$

Solve in terms of x_6

$$
\begin{aligned}
x_1 &= \binom{n}{3} - n(n-2)d + n\binom{d}{2} + nd^2 + \sum_{v\in V}\binom{d(v)}{2} - x_6 \\
x_2 &= n(n-2)d - 2n\binom{d}{2} - 2nd^2 - 2\sum_{v\in V}\binom{d(v)}{2} + 3x_6 \\
x_3 &= n\binom{d}{2} - x_6 \\
x_4 &= nd^2 - x_6 \\
x_5 &= \sum_{v\in V}\binom{d(v)}{2} - x_6
\end{aligned}
$$

$$
\begin{aligned}
x_2 + 3x_3 &= n(n-2)d + n\binom{d}{2} - 2nd^2 - 2\sum_{v\in V}\binom{d(v)}{2} \\
&\leq n(n-2)d - 2nd^2 - n\binom{d}{2} \\
&= \frac{nd(2n-3-5d)}{2}
\end{aligned}
$$

But $x_2 \geq 0$ and $x_3 \geq 0$, so

$$
d \leq \frac{2n-3}{5}
$$

INDUCED 2-PATHS

Thomassé's Conjecture 2006

A digraph on n vertices has at most $\dfrac{n^3}{15} + 0(n^2)$ *induced directed* 2*-paths*

(No condition on degrees or triangles)

In our notation:

$$x_4 \le \frac{n^3}{15} + 0(n^2)$$

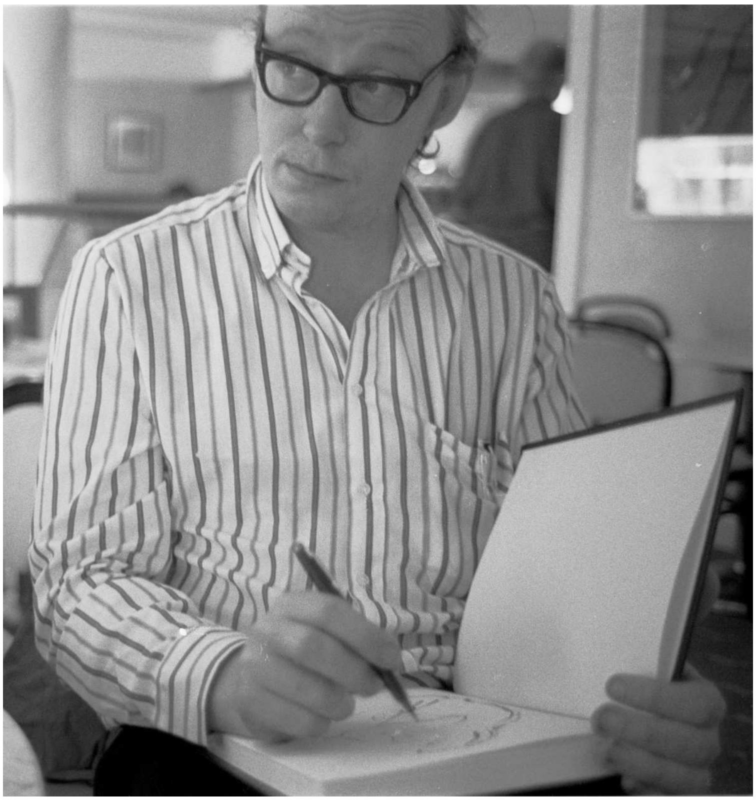
Similar approach to above gives:

$$x_4 \;\le\; \frac{2}{5}x_2 + \frac{1}{10}x_3 + x_4 + \frac{1}{10}x_5 + \frac{9}{5}x_7 \;\le\; \frac{2}{25}n^3$$

Equality:

$$x_1 = \frac{1}{150}n^3,\ x_2 = 0,\ x_3 = 0,\ x_4 = \frac{2}{25}n^3$$

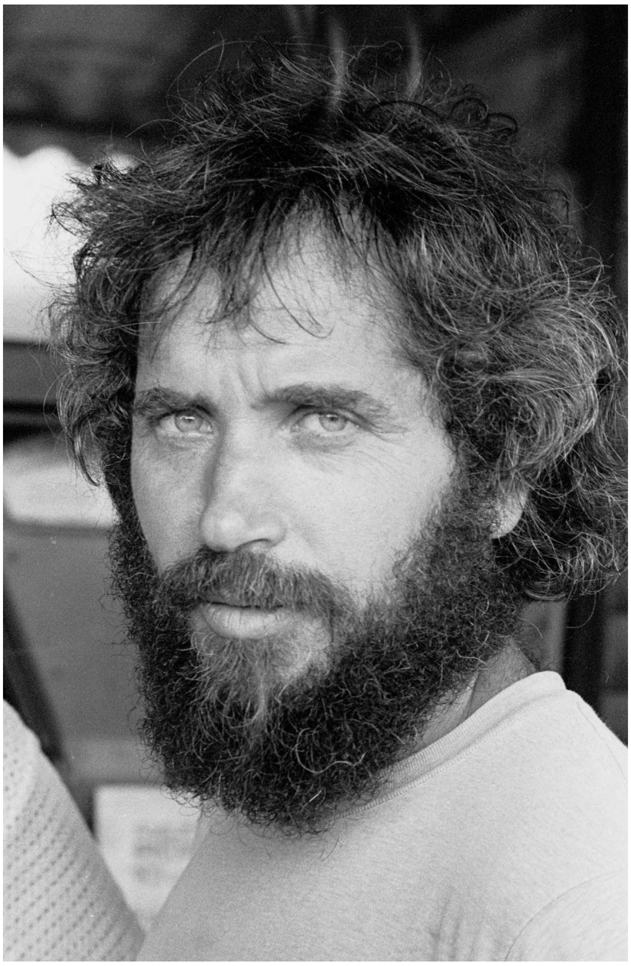
$$x_5 = 0,\ x_6 = \frac{2}{25}n^3,\ x_7 = 0$$



Roland Häggkvist

Paul Seymour

Vašek Chvátal



Endre Szemerédi

Stephan Thomassé

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